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The Existence of a Phase Transition in Classical Antiferromagnetic Models

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For a wide class of antiferromagnetic models we prove the existence of a phase transition using an extended Peierls argument, taking into account a result of Dobrushin [R. L. Dobrushin, *Funct. Anal. Appl.* **2**:44 (1968); in English, **2**:302 (1968)] for an antiferromagnetic Ising model and the results of Malyshev [V. Malyshev, *Comm. Math. Phys.* **40**:75–82 (1975)] for ferromagnetic models (such as the anisotropic rotator). In particular we review a result of Fröhlich, Israel, Lieb, and Simon [J. Fröhlich *et al.*, *J. Stat. Phys.* **22**(3):297–347 (198)] obtained when reflection positivity holds.

KEY WORDS: Phase transitions; antiferromagnetic models.

1. GENERAL STRATEGY

We define a general antiferromagnetic model, which contains, as a particular case, a classical antiferromagnetic anisotropic Heisenberg model with an external magnetic field. In the set of possible spins in each site we fix two particular points (corresponding to spins pointing along the positive and negative z axes). In these points we assume some extremality properties for the interactions. We take some fixed neighborhoods of these points and define a border (block wall) separating the two phases in the lattice. We give an "intrinsic" definition and prove the equivalence to Malyshev's (1) recursive definition (see Appendix). We can then use Malyshev's estimates for the number of possible borders with a given number of sites.

The probability of each border can be estimated comparing the energy

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of a set (cluster) of configurations containing the border with the energy of the cluster of configurations obtained first by erasing the border. The proof of the existence of a phase transition is achieved using standard Peierls arguments.

1.1. Notations. Restrictive Conditions

Let $\mathbb{Z}$ be the group of the integers and $T = \mathbb{Z}^\nu$ (where $\nu \geqslant 2$) the lattice.

Let S be a compact separable metric space with a nonnegative finite measure.

Let $U: S \times S \mapsto R$ be a real measurable function such that (1) $U(S_1, S_2) = U(S_2, S_1)$, $\forall S_1, S_2 \in S$, and (2) U is bounded from below. Let $h: S \mapsto R$ be a real measurable function.

We separate the lattice T into two sublattices R_1 and R_2, in such a way that $T = R_1 \cup R_2$ and such that for every site of R_1 all its nearest neighbors are in R_2 (and conversely).

To simplify the notations we are going to treat the case $\nu = 2$. The generalization to $\nu > 2$ is trivial.

We put $\Delta = \{t \in T: |t| = 1\}$ and take as general Hamiltonian

$$U_V(s_1, \ldots, s_V/\{s(t)\})$$
$$= \sum_{\substack{t_i - t_j \in \Delta \\ t_i, t_j \in V}} U(s_1, s_2)/2 + \sum_{i=1}^{|V|} \sum_{\substack{t \notin V \\ t \in \partial V}} U(s_i, s(t)) + \sum_{i=1}^{|V|} h(s_i) + \eta\psi,$$

where $V = \{t_1, \ldots, t_V\}$ is any finite subset of the lattice T and $s_i \in S$ is the spin in site $t_i \in V$ and $\{s(t)\}$ is a given configuration outside V. ∂V is the set of sites $t \notin V$ such that there exist $t' \in V$ and $t - t' \in \Delta$.

The staggered field ψ is given by

$$\psi = \sum_{i \in R_1} U(s_i, s_0) - \sum_{i \in R_2} U(s_i, s_0)$$

and its mean $\varphi = \langle\psi\rangle_V/|V|$ is the staggered field by site, actually the difference of magnetization on the sublattices.

$s_0 \in S$ is a spin value verifying some conditions (it will be fixed from now on).

It is then known by Dobrushin (4) that, under suitable conditions on U, there exists at least one Gibbs conditional probability distribution associated with the Hamiltonian U_V.

Definition: Admissible Transformation.[1] Let (X_1, Σ_1, μ_1) and (X_2, Σ_2, μ_2) be two measure spaces. A measurable bijection γ from X_1 to X_2 will be called admissible if the measure $\mu_1\gamma^{-1}$ is absolutely continuous with respect to μ_2 and the corresponding Radon–Nykodim derivative satisfies the following conditions for some real constants c_1, c_2 almost everywhere on (X_2, Σ_2, μ_2):

$$0 < c_1 \leqslant \frac{d(\mu_1 \circ \gamma^1)}{d\mu_2} \leqslant c_2 < \infty$$

We restrict ourselves to the U and h verifying the following conditions:

$U1$ $U(s_1, s_2)$ is a continuous function with two absolute minimum in (s_0, s_0') and (s_0', s_0).

$U2$ g symmetry: there exists a continuous bijection of S, say, g, measure preserving such that

$U2.1$ $$gs_0 = s_0'$$

$U2.2$ $$g = g^{-1}$$

$U2.3$ $$U(gs_1, gs_2) = U(s_1, s_2), \qquad \forall s_1, s_2 \in S$$

$U3$: There exist neighborhoods 0_1 and 0_2 of s_0 with $0_1 \subset 0_2$ such that $\mu(0_1) > 0$ and $0_2 \cap g0_2 = \phi$. There exists $\epsilon > 0$ such that

$$U(s_1, s_2) < U(s_1', s_2') - \epsilon, \qquad \forall (s_1, s_2) \in 0_1 \times g0_1$$
$$\forall (s_1', s_2') \notin 0_2 \times g0_2$$

where $A_1 \times A_2$ stands for the Cartesian product.

$U4$: $\mu(0_2 - 0_1) > 0$ and there exists an admissible transformation χ: $0_2 - 0_1 \mapsto A_0 \subset 0_1$, where A_0 is a Borel set that $\mu(A_0) > 0$. χ verifies the following conditions:

$U4.1$: $$\forall s_1 \in g0_1 \quad \forall s_2 \in 0_2 - 0_1: \qquad U(s_1, \chi s_2) < U(s_1, s_2) - \epsilon$$

$U4.2$: $$\forall s_1 \in 0_2 - 0_1 \quad \forall s_2 \in g(0_2 - 0_1):$$
$$U(\chi s_1, g\chi g s_2) < U(s_1, s_2) - \epsilon$$

Remark 1. Let $\bar{\chi} = g\chi g : g(0_2 - 0_1) \mapsto g(0_1)$. This transformation has the same properties as χ, but acts in the neighborhoods $g0_1$ and $g0_2$ of s_0'.

Remark 2. Conditions $U3$ and $U4$ make this model antiferromagnetic.

$U5$: There exists a finite family of sets $F_1, \ldots, F_k$ such that $\mu(F_i) > 0$, $\forall i$, forming a partition of $S - (0_2 \cup g0_2)$. There exists admissible transformations $f_i : F_i \mapsto A_i \subset 0_1$, where A_i are the Borel sets such that $\mu(A_i) > 0$. Let us put

$$b = \min\left\{ \frac{\partial \mu \circ f_i}{\partial \mu}, \frac{\partial \mu \circ \chi}{\partial \mu} \right\}, \qquad b > 0.$$

In the following ϵ plays a similar role to the "interaction magnitude," $|I|$ on the Ising model. These conditions are the adaptation of Malyshev's conditions to the ferromagnetic case. Owing to the existence of the magnetic field, we have to introduce a shift for removing a border (following Dobrushin (2)); therefore as we must compare the energy variation in some remote sites we have to introduce a new condition $U6$ (easily verified in the Heisenberg model).

$U6$: There exists $d > 0$ such that

$U6.1$:
$$d = \sup_{\substack{s_1, s_3 \in 0_1 \\ s_2, s_4 \in g0_1}} \left[U(s_1, s_2) - U(s_3, s_4) \right]$$

$U6.2$:
$$\sup_{\substack{s_1 \in 0_1 \\ s_2 \in g0_1}} U(s_1, s_2) < \inf_{\substack{s_3 \in 0_1 \\ s_4 \in 0_1}} U(s_3, s_4) + 3d - \epsilon$$

$H1$: $h : S \mapsto R$ is a bounded function, i.e., there exists a real number Δh such that

$$\Delta h = \sup_{s \in S} h'(s) - \inf_{s \in S} h(s)$$

This condition is sufficient to prove the desired result; it means that the magnetic field must be bounded in some sense (cf. Ising model, Dobrushin[(2)]; see also Ref 6).

1.2. Theorem

If U verifies conditions $U1$–$U6$ and h verifies condition $H1$, and if $\epsilon > \Delta h$, then there exist at least two different Gibbs states for sufficiently small temperature $T = 1/\beta$.

Remark 3. The result (and the proof) extend to noncompletely translation invariant models. In fact, it is sufficient to have translation invariance in a single lattice axis. In particular, we can handle a "staggered" field in the other $\nu - 1$ lattice axis.

Owing to possible nontranslation invariance we prove the result only for $\epsilon > \Delta h$. In simpler cases, we can have the result in some less restricted regions; for example, for the Ising antiferromagnetic model Dobrushin proved the existence of a phase transition when $\Delta h < 2\epsilon$, choosing shift direction in a suitable way (see Ref 2). In Fig. 1 one sketches a staggered magnetic field in planes parallel to $0XY$. We can handle this case if h and U are independent of z. Dobrushin shift must be done with fixed X and Y.

We want to prove that φ is a discontinuous function of η at $\eta = 0$ in the thermodynamic limit and at sufficiently low temperatures. In this limit ψ is an antisymmetric nondecreasing function of η, so it suffices to show that

$$\hat{\varphi} = \lim_{\eta \to 0^+} \varphi(\eta) > 0$$

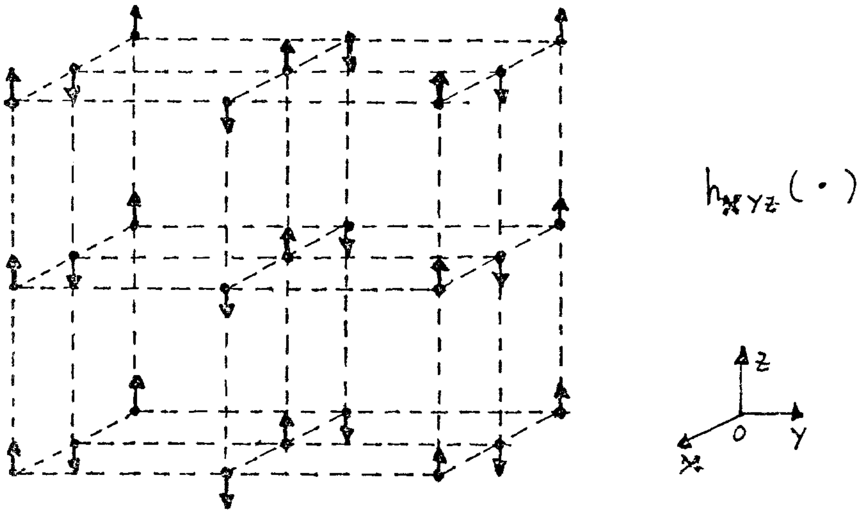

Fig. 1.

which is certainly the case if we can use special boundary conditions obtaining a lower bound $r > 0$ for φ_V independent of $|V|$.

2. PROOF OF THE THEOREM

Let us fix a site $t_0 \in R_1 \cap V$ (or equivalently a site $t_0' \in R_2 \cap V$). Our aim is to prove that the probability that $s_{t_0} \in S - g0_1$ (or $s'_{t_0} \in S \equiv g0_1$) is less than $1/2$; in this case using "g symmetry" the theorem follows.

Definition: Interacting Sites and Sets. We say that $t_1, t_2 \in T$ are *interacting sites* if $|t_1 - t_2| = 1$. In a similar way sets $T_1, T_2 \subset T$ will be said to be *interacting* if there exist interacting sites $t_1 \in T_1$ and $t_2 \in T_2$.

Definition: d-Connection. An ordered set $\{t_1, t_1, \ldots, t_n\} \subset T$ is a d *path connecting* t_1 to t_n if $|t_i - t_{i+1}| \leqslant d$ for $i = 1, 2, \ldots, n-1$. A set $T' \subset T$ is d connected if for every $t', t'' \in T'$ there exists a d path connecting t' to t''.

Definition: A Site. Let $A \subset S$ be a measurable set such that $\mu(A) > 0$. A site $t_0 \in T$ is said to be an A site of a given configuration $\{s(t)\}$ if $s(t_0) \in A$.

Definition. In a given configuration, a site t will be called *well oriented* (W) (i) if t is a 0_1 site in R_1, or (ii) if t is a $g0_1$ site in R_2.

The sites following neither of these conditions will be generally called bad oriented. Among those, the sites t such that $g \circ s(t)$ is well oriented, i.e., (i) if t is a $g0_1$ site in R_1, or (ii) if t is a 0_1 site in R_2, will be called *very bad oriented* (B). The bad-oriented sites t not very-bad-oriented will be called intermediate (b).

In the sequel we take only configurations with all sites outside V well oriented (*boundary conditions*).

Now, with these definitions we shall prove that the probability of a bad-oriented site in V is less than $1/2$.

Let t_0 be a bad-oriented site on V and let R_b be the maximal 1-connected component of bad-oriented-sites (b or B) which contain t_0 ($R_b \subset V$). $T \backslash R_b$ is the union of its 1-connected components $R_0, R_1, \ldots, R_q$. The sites of these components interacting with R_b are W. Let us call R_0 the component which contains $T \backslash V$.

2.1. "Intrinsic" Definition $R_b(B)$ "outer B border of R_b"

$R_b(B)$ is the subset of R_b which contains all the sites that can be joined to $\mathscr{R}_0$ by a 1-path not containing two consecutive B sites. (This definition is equivalent to Malyshev's definition of contours; see Refs. 1 and 7 and Appendix.) Let us denote by $\mathscr{R}_0, \mathscr{R}_{i_1}, \ldots, \mathscr{R}_{i_m}$ those $\mathscr{R}_i$ which interact with $R_b(B)$ and by $R_{j_1}, \ldots, R_{j_k}$ the others. We put

$$\mathscr{R}_b^0 = \mathscr{R}_b \cup \mathscr{R}_{j_1} \cup \cdots \cup \mathscr{R}_{j_k}$$

Lemma 1: A Geometric Estimation.[1] "The number of possible sets $R_b(B)$ consisting of exactly l sites does not exceed $cl^2\gamma^l$, for some constant c."

Proof. We are going to sketch the nonstandard part of the proof, that is, we show that $R_b(B)$ is $\sqrt{2}$ connected. The proof is the same as in Ref. 1.

We remark that simple geometric considerations show that the set Q_i consisting of the sites of $R_b(B)$ interacting with a $\mathscr{R}_i$ is $\sqrt{2}$ connected. (Q_i can be seen as a closed $\sqrt{2}$ path $t_{i_0}, t_{i_1}, \ldots, t_{i_l}$ where $t_{i_j} \in Q_i \subset R_b(B)$, and $t_{i_0} = t_{i_l}$.)

For all pairs of sites $m, n \in R_b(B)$ we are going to construct a $\sqrt{2}$ path on $R_b(B)$ (see Fig. 2). The definition of $R_b(B)$ says that there exists a 1-path from m (resp. n) to a site $\overline{m}$ (resp. $\overline{n}$) interacting with $\mathscr{R}_0$. In these paths there are eventually sites belonging to some $\mathscr{R}_i$, say, $i_1, \ldots, i_k$, where i_1 interacts with a t_{i_α} and i_k interacts with a t_{i_β} $\alpha, \beta \in \{1, 2, \ldots, l\}$. We have a $\sqrt{2}$ path belonging to $R_b(B)$ if we replace the piece of path $i_1, \ldots, i_k$, by $t_{i_\alpha}, \ldots, t_{i_\beta}$ a subpath of $t_{i_1}, \ldots, t_{i_l}$ and therefore avoiding $\mathscr{R}_i$. ■

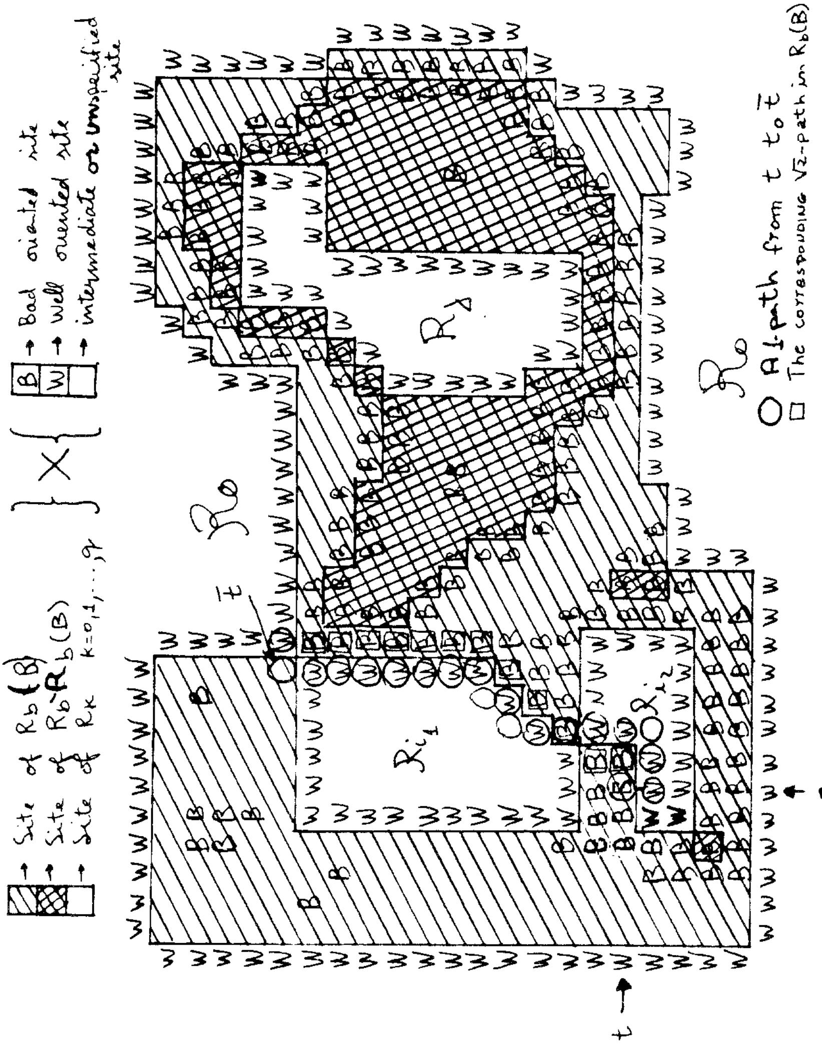
Site of $R_b(B)$
Site of $R_b - R_b(B)$
Site of R_k $k=0,1,\ldots,q$
B → Bad oriented site
W → Well oriented site
→ intermediate or unspecified site
○ A $\sqrt{2}$-path from t to $\bar{t}$
□ The corresponding $\sqrt{2}$-path in $R_b(B)$

Fig. 2.

2.2. Probability Estimate of a Given $R_b(B)$ Containing l Sites

We are going to define into two steps G_1 and G_2 a transformation $G = G_1 \circ G_2$ of the configuration space into itself.

$G1$: *Dobrushin's Transformation (Shift).* Let $t \in T$, and let $\hat{t}$ be the site just below t (in some fixed axis). Let $\check{t}$ be the site just above t (in the same axis). We define G_1 by

$$G_1(s(t)) = \begin{cases} s(t) & \text{if} \quad t \notin R_b^0 \\ s(\hat{t}) & \text{if} \quad t \text{ and } \hat{t} \in R_b^0 \\ gs(t) & \text{if} \quad t \in R_b^0 \quad \text{and} \quad \hat{t} \notin R_b^0 \end{cases}$$

This is a generalization of Dobrushin's transformation to a continuous case (compare, for example, with Griffiths[6]).

Definition. Let $A \subset T$; we define $\overline{A}^V$ by

$$\overline{A}^V = \{t \in T : \hat{t} \in A\}$$

We know that $R_b(B) = R_b^0 \backslash \{[R_b \backslash R_b(B)] \cup R_{j_1} \cup \cdots \cup R_{j_m}\}$. The transformation G_1 *leads us to take the set*

$$\tilde{R}_b(B) = R_b^0 \backslash \overline{\{[R_b \backslash R_b(B)] \cup R_{j_1} \cup \cdots \cup R_{j_m}\}}^V$$

Two elementary, but useful in the following, properties of $\tilde{R}_b(B)$ are collected in the following lemma:

Lemma 2. (2.1) $\tilde{R}_b(B)$ has the same number of sites as $R_b(B)$; (2.2) In the configuration $G_l \circ s(t)$ the $\tilde{R}_b(B)$'s spins only have types b or W.

Proof. See Ref 7. ■

Remark. It is trivial to see that

$$\tilde{R}_b(B) = \{t \in R_b : \hat{t} \in R_b(B)\} \cup \{t \in R_b(B) : \hat{t} \in R_0 \cup R_{i_1} \cup \cdots \cup R_{i_k}\}$$

$G2$: *Malyshev's Transformation.* We are able now to define a second transformation G_2 which will take spins from $R_1 \cap \tilde{R}_b(B)$ to 0_1 and from $R_2 \cap \tilde{R}_b(B)$ to $g0_2$ [i.e., making all $\tilde{R}_b(B)$ spins well oriented] using the transformations χ, $\bar{\chi}$, and f_i, $i = 1, \ldots, k$ (these defined in condition $U5$).

We define G_2, a transformation of space of configurations into itself by (compare with Ref. 1)

$$G_2 \circ s(t) = \begin{cases} f \circ s(t) & \text{if} \quad t \in \tilde{R}_b(B) \\ s(t) & \text{if not} \end{cases}$$

where f is defined as

$$f \circ s(t) = \begin{cases} \left.\begin{array}{lll} s(t) & \text{if} & s(t) \in 0_1 \\ \chi s(t) & \text{if} & s(t) \in 0_2 - 0_1 \\ f_i \circ s(t) & \text{if} & s(t) \in F_i \\ \chi g s(t) & \text{if} & s(t) \in g(0_2 - 0_1) \end{array}\right\} & \text{and} \quad t \in R_1 \\ \left.\begin{array}{lll} s(t) & \text{if} & s(t) \in g0_1 \\ g\chi g s(t) & \text{if} & s(t) \in g(0_2 - 0_1) \\ g \circ f_i s(t) & \text{if} & s(t) \in F_i \\ g\chi s(t) & \text{if} & s(t) \in 0_2 - 0_1 \end{array}\right\} & \text{and} \quad t \in R_2 \end{cases}$$

We have seen that $R_b(B)$ was separated from its complement in T by two kinds of borders: (1) the border separating $R_b(B)$ from $R_b \backslash R_b(B)$; (2) the border separating $R_b(B)$ from $\mathscr{R}_0 \cup \mathscr{R}_{i_1} \cup \cdots \cup \mathscr{R}_{ik}$. In the first case across the border the interactions of $\{s(t)\}$ are B–B and in the second case B–W or b–w.

On the contrary $\tilde{R}_b(B)$ is separated from its complement by the following kinds of borders: (1) the border separating $\tilde{R}_b(B)$ from $\overline{R_b \backslash R_b(B)}^V$; (2) the border separating $R_b(B)$ from $R_0 \cup R_{i_1} \cup \cdots \cup R_{i_k}$. In these two cases across the borders all the interactions of $\{G_2 \circ G_1 \circ S(t)\}$ are W–W, that is the transformation $G = G_2 \circ G_1$ erases the border $R_b(B)$.

Let us decompose the set of configurations into several subsets (clusters). We consider partition $\mathscr{P}$ of S with subsets $F_1, F_2, \ldots, F_k$, $g0_1$, $g(0_2 - 0_1)$, $0_2 - 0_1$, and 0_1; two configurations $\{s_1(t)\}$ and $\{s_2(t)\}$ belong to the same cluster if $\forall t_0 \in R_b(B)$, $s_1(t)$ and $s_2(t_0)$ belong to the same element of the partition $\mathscr{P}$. In this case we write $s_1 \sim s_2$. The number of possible clusters does not exceed $(k+3)^l$, where l is the number of sites in $R_b(B)$.

Estimation of Energy Variation. Let us denote by $U_s = U_V(s(t_1), \ldots, s(t_V)/\{s(t)\})$ the energy of a configuration $\{s(t)\}$ belonging to a given cluster L. In U_s we have three terms: a term of internal interaction in V; another term between V and $T - V$; finally a term involving the magnetic field. As we take $\eta = 0$ there are no other terms.

Lemma 4. For every configuration $\{s(t)\}$ with boundary conditions as above, we have

$$U_{Gs} < U_s - (\epsilon - \Delta h)l$$

where ϵ and Δh are defined in conditions U_1–U_6 and H_1.

Proof. As we want an estimate of difference $U_s - U_{Gs}$, it is easy to see that we can replace V by $R_b(B)$ in internal and border energy of U_s and V by $\tilde{R}_b(B)$ in U_{Gs}. By commodity we are going to define a natural bijection $\sim$ between $R_b(B)$ and $\tilde{R}_b(B)$. Let us put

$$\check{t}^n = \left.\check{\vdots\,\check{t}}\right\}n \text{ times} \qquad \text{and} \qquad \hat{t}^n = \left.\hat{\vdots\,\hat{t}}\right\}n \text{ times}$$

Now we define $\sim$:

$$t \in R_b(B) \to \tilde{t} \in \tilde{R}_b(B)$$

If there exists $n > 0$ such that $\check{t}^n \in \tilde{R}_b(B)$ then we take the smallest one, n_0, and put $\tilde{t} = \check{t}^{n_0}$ If not, we put $\tilde{t} = \hat{t}^{m_0}$, where m_0 is the greatest m such that $\hat{t}^m \in R_b(B)$; this case appears only if $\check{t} \in R_0$. For each $t \in R_b(B)$ we take the sum of the four terms $U(s(t), s(t_i))$, $t_i \in \Delta + t$; we compare these terms with the corresponding terms for $\tilde{t} \in \tilde{R}_b(B)$.

1st case: $t + \Delta \in R_b(B)$. $G \circ s(\tilde{t}_i)$ is always W; as $s(t_i)$ is b or B conditions U_3 and U_4 give

$$U\big(G \circ s(\tilde{t}_i), G \circ s(\tilde{t})\big) < U\big(s(t_i), s(t)\big) - \epsilon \tag{*}$$

except when $s(t)$ and $s(t_i)$ are simultaneously B; in this case we have $s(t) = G \circ s(\tilde{t})$ and therefore

$$U\big(G \circ s(\tilde{t}), G \circ s(\tilde{t}_i)\big) = U\big(s(t), s(t_i)\big)$$

However, for every t which is a B site, there exists from the definition of $R_b(B)$ at least a $t_i \in t + \Delta$, such that $s(t_i)$ is a b site and then verifying (*).

Therefore we have in this case

$$\sum_{t_i \in t + \Delta} U\big(G \circ s(\tilde{t}), G \circ s(\tilde{t}_i)\big) - U\big(s(t), s(t_i)\big) < -\epsilon \tag{**}$$

2nd case: $(t + \Delta) \cap [R_b \backslash R_b(B)] \neq \phi$, but $(t + \Delta) \cap [R_0 \cup R_{i_1} \cup \cdots \cup R_{i_k}] = \phi$. In this case, $s(t)$ is a B site, as well as $s(t_i)$ with $t_i \in t + \Delta$ and $t_i \in R_b \backslash R_b(B)$. The same reasons as in the first case give (**).

3rd Case: $(t + \Delta) \cap [R_b \backslash R_b(B)] = \phi$, but $(t + \Delta) \cap (\mathscr{R}_0 \cup \mathscr{R}_{i_1} \cup \cdots \cup R_{i_k}) \neq \phi$. If $\tilde{t} = \check{t}$ we have (**) as in the first case. If $t_i \in (R_0 \cup \cdots \cup R_{i_k})$, then t_i is a w site and t a B or b site. G transforms these W–B or W–b interactions in W–W interactions. We find again (*) even if $\tilde{t}$ interacts with $\check{t}_i$ and generally

$$G \circ s(\check{t}_i) = s(\check{t}_i) \neq s(t_i)$$

On the contrary, if $\tilde{t} \neq \check{t}$, we know that $s(t)$ is b or B; as $\hat{t} \in R_0 \cup R_{i_1} \cup \cdots \cup R_{i_k}$ we know that $s(\hat{t})$ is a W site and from conditions $U3$ and $U4$ this is the worst case. As $s(\tilde{t})$ and $s(\check{t})$ are W sites the worst case appears when all other spins are in B. In these cases, three B–B interactions and a W–B interaction became (via $\sim$ and G transformations) four W–W interactions. Condition $U_{6.1}$ ensures that we have lost at most d for each W–W interaction; condition $U_{6.2}$ ensures a gain of at least $\epsilon + 3d$ in interaction B–W. Therefore we find again (**).

4th case: $t + \Delta \cap [R_b \backslash R_b(B)] \neq \phi$, and $t + \Delta \cap (R_0 \cup \mathscr{R}_{i_1} \cup \cdots \cup \mathscr{R}_{i_k}) \neq \phi$. This case is treated essentially as the former. We conclude then that the first term in (**) is less than $-l\epsilon$, i.e., a decrease of ϵ by site.

To allow for magnetic field terms we must add quantity Δh by site, and then we obtain $-l(\epsilon - \Delta h)$ as a majorant of $U_{Gs} - U_s$.

Remark 1. If $\Delta h < \epsilon$ the energy decreases. In Ising model we have, as similar conditions, $|H| < 4|\mathscr{I}|$, where H is the magnetic field and $\mathscr{I}$ the coupling constant (cf. [6]).

Remark 2. We are forced to restrict transformations to $R_b(B)$ because to transform all $\mathscr{R}_0$ complement would introduce a volume term in the energy estimates due to the existence of magnetic field.

As g is measure preserving and all other transforms are admissible, the l integrations give each one a factor b. Note that not all the transformations are measure preserving but the admissibility implies this estimate. Therefore we have

$$Z_L / Z_{GL} \leqslant \exp\{-\beta l(\epsilon - \Delta h)\} b^{-l}$$

Putting together all the estimates we control the probability for t_0 to be a bad-oriented site (b or B). We write these estimates in a schematic way as in the usual Peierls argument with the difference that continuous spins oblige us to introduce the concept of cluster of configurations. Then we have

$$\begin{aligned}
&P(t \text{ is bad-oriented}) \\
&\quad \leqslant P(t_0 \text{ is in some border's interior}) \\
&\quad \leqslant \sum_l P(t_0 \text{ is interior to a } l\text{-sites border}) \\
&\quad \leqslant \sum_l \text{ (number of geometric border with } l\text{-sites)} \\
&\qquad \times / P(\text{each } l \text{ sites border}) \\
&\quad \leqslant \sum_l c l^2 \gamma^l \text{ (Number of energy cells: “clusters”)} \\
&\qquad \times P(\text{cell of greater probability}) \\
&\quad \leqslant \sum_l c l^2 \gamma^l \times (k+3)^l \exp\{-\beta l(\epsilon - \Delta h)\} b^l
\end{aligned}$$

This estimate is uniform in V. Choosing β sufficiently great (temperature sufficiently low), the last term can be as small as we wish, and in particular smaller than $1/2$ for β greater than some β_c. This concludes the proof. ■

3. HEISENBERG MODEL

In this model we have

$$S = \left\{ s = (u, y, z) : u^2 + y^2 + z^2 = 1 \right\}$$

and

$$U(s_1, s_2) = |\mathscr{I}|(z_1 z_2 + \alpha(u_1 u_2 + y_1 y_2))$$

where $|\mathscr{I}| \in |\mathbb{R}$ and $|\alpha| < 1$

$$h(s) = h_u u + h_y y + h_z z, \qquad \text{where} \quad h = (h_u, h_y, h_z) \in \mathbb{R}^3;$$

and

$$\Psi = \sum_{i \in R_1} z_i - \sum_{i \in R_2} z_i$$

U and f verify (see Refs. 1 and 7) conditions $U1$–$U6$ and $F1$. Then we have the following:

Corollary. The classical Heisenberg antiferromagnetic with arbitrary parameter of anisotropy and with magnetic field bounded $|h| < |\mathscr{I}|/2$ has a phase transition at sufficiently low temperatures.

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APPENDIX

Malyshev Definition (inductive), see Ref. 1: $R_b(B)$ consists of, and only of the following sites belonging to R_b:

1st step: the sites which interact with $\mathscr{R}_0$.

2nd step: the sites which are not B sites and interacting with at least one site of $R_b(B)$ defined before.

3rd step: B sites interacting with at least one site of $R_b(B)$ defined earlier, which is not a B site.

4th step: the sites interacting with those $\mathscr{R}_i$ which contains at least one site interacting with $R_b(B)$ defined earlier.

We return to step 2 and we continue this procedure, until all the sets defined in all these steps are empty.

Lemma. Malyshev's border is equivalent to the "intrinsic" border.

Proof. Let X be the Malyshev border and let Y be the "intrinsic" border.

(a) $$t \in X \Rightarrow t \in Y$$

The sites belonging to X are defined into four steps.

1st step: A site t defined in the first step has trivially a 1-path joining it to $\mathscr{R}_0$. This path has only a site, t. Therefore there are not two consecutive B sites in this path.

For the sites t, defined in steps 2 or 3, it is sufficient to prove that if all sites defined earlier are connected to $\mathscr{R}_0$ by a 1-path without two consecutive B sites, then the path reunion of $\{t\}$ with this path has the same properties.

2nd step: Sites defined in this step are b sites interacting with a site that is connected to $\mathscr{R}_0$ by a 1-path $t_1, t_2, \ldots, t_n$ without two B sites consecutive. Then the 1-path $t, t_1, t_2, \ldots, t_n$ has the same properties.

3rd step: The sites defined in this step are B, but they interact with a b site t_1, which is connected to $\mathscr{R}_0$ by a 1-path $t_1, t_2, \ldots, t_n$ without two B sites consecutive. Then the path $t, t_1, t_2, \ldots, t_n$ has the same properties.

4th step: The sites t defined in this step interact with a site of a $\mathscr{R}_i$; then t is connected by a 1-path to a site of $R_b(B)$ defined earlier and therefore connected to R_0 by a path with the same properties.

(b) $$t \in Y \Rightarrow t \in X$$

Let $t_1, t_2, \ldots, t_n, t_{n+1}; t_{n+1} = t$ be a 1-path connecting t to $\mathscr{R}_0$ without two consecutive B sites. The idea of this proof is to verify that $t_i \in R_b(B)$ or to some appropriate $R_j t_1 \in X$ because it interacts with R_0.

1st case: We suppose that there are not w sites in the path, i.e., all sites are b or B. We are going to show that if $t_i \in X$ then $t_{i+1} \in X$ and therefore $t \in X$. If t_{i+1} is a B site then t_i (path condition) is b. Therefore t_{i+1} will be included in $R_b(B)$ at the third step consecutive to the t_i inclusion. If t_{i+1} is a b site it will be included by the action of the second step consecutive to the t_i inclusion.

2nd case: There are W sites in the path. Let t_j be the first W site in the path. As $t_j \notin R_b$, t_j must belong to some $\mathscr{R}_i$. The first case proves that $t_{j-1} \in R_b(B)$. Let t_k be the first site after t_j belonging to R_b. t_k interacts with $t_{k-1} \in \mathscr{R}_i$. Then $t_k \in X$ because (4th step) t_k interacts with $t_{k-1} \in R_i$ and R_i interacts with $t_{j-1} \in R_b(B)$.

We achieve the proof of the lemma alternating these two arguments for sites $t_{k+1}, \ldots, t_n$. ■

REFERENCES

1. V. Malyshev, *Comm. Math. Phys.* **40**:75–82 (1975).
2. R. L. Dobrushin, *Funct. Anal. Appl* **2**:44 (1968); English trans., **2**:302 (1968).
3. J. Fröhlich, R. Israel, E. Lieb, and B. Simon, *J. Stat. Phys.* **22**(3):297–347 (1980).
4. R. L. Dobrushin, *Teor. Veroyat. Prim.* **15**:469–497 (1970).
5. J. Ginibre, Systèmes à un nombre infini de degrés de liberté, *CNRS 1970*, 163–175.
6. R. Griffiths, Phase transitions and critical phenomena, *Domb and Green*, Vol. I, Sect. V, 55–70 (1972).
7. F. Calheiros, Systèmes de spin vectoriel: existence d'une transition de phase pour le modèle d'Heisenberg antiferromagnetique et anisotrope, Thèse de 3ème Cycle, CPT-CNRS, Marseille (1980).